



Prescribing geodesics and a variational problem for Riemannian metrics

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ABSTRACT. Given a prescription of unparametrised paths on a manifold M , one path for each tangent direction, we may ask whether these paths agree with the geodesics of a Riemannian metric on M . Generically, this is not the case. Motivated by this fact, we introduce a non-negative functional $\mathcal{E}$ on the space of Riemannian metrics on M so that $\mathcal{E}(g) = 0$ if and only if the geodesics of the metric g agree with the prescribed paths. We compute the variational equations for $\mathcal{E}$ and show that the conformal variational equation is, perhaps surprisingly, of Yamabe type. This allows us to obtain existence results for conformally critical points of $\mathcal{E}$. In particular, in the surface case, every conformal class contains a conformally critical metric, unique up to homothety. As a by-product, we establish that the Blaschke metric of a properly convex projective surface is a critical point for $\mathcal{E}$.

1. Introduction

A prescription of unparametrised paths on a manifold M , one path for each tangent direction, specifies the geometric shape of a family of curves but not the speed at which one travels along them. It is natural to require these paths to be the geodesics of a torsion-free connection on TM , since the geodesics of a Riemannian metric are precisely those of its Levi-Civita connection. Such a prescription is encoded by a *projective structure*: an equivalence class $\mathfrak{p}$ of torsion-free connections on TM , where two such connections are equivalent if they have the same geodesics up to parametrisation. Here by a geodesic for $\nabla \in \mathfrak{p}$ we mean an immersed curve $\gamma : I \rightarrow M$ so that $\nabla_{\dot{\gamma}} \dot{\gamma}$ vanishes identically. A projective structure $\mathfrak{p}$ is called *metrisable* if there exists a Riemannian metric g on M whose Levi-Civita connection ${}^g\nabla$ is an element of $\mathfrak{p}$. A generic projective structure is not metrisable, and necessary and sufficient conditions for metrisability of a two-dimensional projective structure were obtained by Bryant, Dunajski and Eastwood [BDE09].

A manifold M equipped with a projective structure will be called a *projective manifold*. For a closed oriented projective manifold $(M, \mathfrak{p})$ of dimension $n \geq 2$, we introduce a functional $\mathcal{E}$ on the space $\text{Riem}(M)$ of Riemannian metrics on M as follows. Given a representative $\nabla \in \mathfrak{p}$, the difference $\nabla - {}^g\nabla$ is a 1-form on M with values in the endomorphisms of TM ; we write $(\cdot)_0$ for the part of such a 1-form taking values in the trace-free

endomorphisms of TM and define

$$\varepsilon : \text{Riem}(M) \rightarrow \mathbb{R}, \quad g \mapsto \text{Vol}_M(g)^{(2-n)/n} \int_M |(\nabla - {}^g\nabla)_0|^2_g d\mu_g,$$

where $d\mu_g$ denotes the volume form of g , $\text{Vol}_M(g)$ the volume of M with respect to $d\mu_g$ and $|\cdot|_g$ the point-wise norm induced by g on the fibres of $S^2(T^*M) \otimes TM$. It turns out that the non-negative functional ε depends only on the projective equivalence class of ∇ and, moreover, that $\varepsilon(g) = 0$ for $g \in \text{Riem}(M)$ if and only if $\mathfrak{p}$ is metrisable by the metric g . The presence of the factor $\text{Vol}_M(g)^{(2-n)/n}$ makes ε invariant under homothety, that is, $\varepsilon(cg) = \varepsilon(g)$ for all positive constants c and all metrics $g \in \text{Riem}(M)$.

Setting $\Phi_g := (\nabla - {}^g\nabla)_0$, the variational equations for ε take the form (see Proposition 3.1 for details)

$$T_g(\Phi_g) - 2\ell_g^*(\Phi_g) = kg,$$

where k is a constant and $T_g(\Phi_g) \in \Gamma(S^2(T^*M))$ is a purely algebraic term in Φ_g . The map ℓ_g^* is the formal adjoint of a first-order differential operator ℓ_g obtained from the linearisation of the map $g \mapsto {}^g\nabla$, which assigns to a metric its Levi-Civita connection.

An important class of (in general) non-metrisable projective surfaces carrying a distinguished metric – the so-called *Blaschke metric* – is given by the properly convex projective surfaces. They are of particular interest due to their connection to higher Teichmüller theory. For background we refer to the landmark paper by Hitchin [Hit92]; see also [Wie18] for a recent survey. Given a Riemann surface structure on M , every holomorphic cubic differential C determines a Riemannian metric g – the Blaschke metric – via Hitchin’s self-duality equations [Hit87], and the pair (g, C) encodes a properly convex projective structure on M [Lof01, Lab07]. Alternatively, the Blaschke metric arises via a hyperbolic affine sphere over the universal cover of M , obtained from a Monge–Ampère equation solved by Cheng–Yau [CY77, CY86]; see [Lof10] for a survey. Our first main result complements these pictures by relating the Blaschke metric to a variational principle intrinsic to the projective structure:

Theorem A. *Let $(M, \mathfrak{p})$ be a closed oriented properly convex projective surface of negative Euler characteristic. Then the Blaschke metric of $\mathfrak{p}$ is a critical point for ε .*

We now turn to conformal variations of ε . In two dimensions, solving the variational equations for conformal variations is equivalent to solving a Poisson equation, and this allows us to prove:

Theorem B. *Let $(M, \mathfrak{p})$ be a closed oriented projective surface. Then every conformal structure on M contains a conformally critical metric, unique up to homothety.*

A surprising aspect of ε is its close connection to the Yamabe equation [Yam60] and to the Einstein–Hilbert functional, which assigns to a metric g its volume-normalised total scalar curvature $\text{Vol}_M(g)^{(2-n)/n} \int_M S_g d\mu_g$. To

explain this connection, we use [Met20, Thm. 2.4], which shows that the choice of a conformal structure $[g]$ on a projective manifold $(M, \mathfrak{p})$ distinguishes a unique torsion-free connection $^{[g]}\nabla$ preserving $[g]$ – a so-called *Weyl connection* – and a 1-form $A_{[g]}$ on M with values in the endomorphisms of TM so that $^{[g]}\nabla + A_{[g]}$ lies in $\mathfrak{p}$.

For a metric g on $(M, \mathfrak{p})$, we define the *conformal defect* as

$$V_g = \frac{(n+1)(n-2)}{(n+2)} |A_{[g]}|_g^2 - \operatorname{tr}_g \operatorname{Ric} \left(^{[g]}\nabla \right),$$

where $\operatorname{Ric} \left(^{[g]}\nabla \right)$ denotes the Ricci curvature of $^{[g]}\nabla$. The motivation for introducing V_g is its relevance in finding conformally critical metrics for $\mathcal{E}$. Recall that, for $n \geq 3$ the conformal Laplacian of g is defined as

$$L_g := 4 \left(\frac{n-1}{n-2} \right) \Delta_g + S_g.$$

Using V_g , we define the *projective-conformal Laplacian* $\mathcal{L}_g = L_g + V_g$ and show that finding a metric in $[g]$ that is conformally critical for $\mathcal{E}$ amounts to solving an equation of Yamabe type

$$\mathcal{L}_g u = C u^{(n+2)/(n-2)}$$

for some constant C and smooth positive function u on M .

Under a conformal change $g \mapsto \tilde{g} = \exp(2f)g$, the conformal defect changes as $V_{\tilde{g}} = \exp(-2f)V_g$. As a consequence of this, it follows that the *projective scalar curvature*

$$S_g := S_g + V_g$$

transforms exactly as the scalar curvature under a conformal change $g \mapsto \tilde{g}$. Furthermore, we have the identity

$$\mathcal{E}(g) = \frac{n+2}{(n+1)(n-2)} \operatorname{Vol}_M(g)^{(2-n)/n} \int_M S_g d\mu_g,$$

showing that our functional closely resembles the Einstein–Hilbert functional. We define the *conformal locus* of $[g]$ as $\Sigma_{[g]} := \{p \in M \mid V_g(p) = 0\}$. Note that $\Sigma_{[g]}$ is a conformal invariant of $[g]$. Building on the solution [Tru68, Aub76, Sch84] of the Yamabe problem [Yam60] and the theory of elliptic equations of Yamabe type (see in particular [DH05]), we obtain:

Theorem C. *Let $(M, \mathfrak{p})$ be a closed oriented projective manifold of dimension $n \geq 3$ and $[g]$ a conformal structure on M . Suppose that either $\Sigma_{[g]} = M$, or that $n \geq 4$ and there exists $p_0 \in M$ with $V_{\tilde{g}}(p_0) < 0$ for some (and hence any) $\tilde{g} \in [g]$. Then $[g]$ contains a conformally critical metric for $\mathcal{E}$.*

Drawing further on the analogy with the Yamabe equation, it is tempting to speculate that on a closed oriented projective manifold $(M, \mathfrak{p})$, every conformal equivalence class $[g]$ contains a conformally critical metric for $\mathcal{E}$. Some of the cases left open by Theorem C appear to be within reach of the localisation theory for elliptic equations of Yamabe type developed by Druet and Hebey [DH05]. Others – most notably the borderline regime in which $V_g \geq 0$ with $\Sigma_{[g]} \neq \emptyset$ – parallel the situation in which the classical Yamabe problem was settled by Schoen [Sch84] via the positive mass theorem of

Schoen and Yau [SY79, SY81]. We hope to return to these unresolved cases in future work.

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2. Setup

2.1. Projective structures

Let M be a closed oriented smooth n -manifold with $n \geq 2$. Recall that the difference of two torsion-free connections ∇ and ∇' on TM is a section of the vector bundle $E = S^2(T^*M) \otimes TM$. For a section $\Phi = (\Phi_{jk}^i)$ of E we define $\text{Con}(\Phi) \in \Omega^1$ by the pointwise tensor contraction

$$[\text{Con}(\Phi)]_i = \Phi_{ik}^k.$$

For $\alpha = (\alpha_i) \in \Omega^1$ we obtain a section $\text{Sym}(\alpha)$ of E by the rule

$$[\text{Sym}(\alpha)]_{jk}^i = \delta_j^i \alpha_k + \delta_k^i \alpha_j.$$

Observe that $\text{Con}(\text{Sym}(\alpha)) = (n+1)\alpha$.

For a section Φ of E we denote by Φ_0 its part in the kernel of the above contraction, that is,

$$(2.1) \quad \Phi_0 := \Phi - \frac{1}{n+1} \text{Sym}(\text{Con}(\Phi)).$$

In particular, we have

$$(\text{Sym}(\alpha))_0 = \text{Sym}(\alpha) - \frac{1}{(n+1)} \text{Sym}(\text{Con}(\text{Sym}(\alpha))) = \text{Sym}(\alpha) - \text{Sym}(\alpha) = 0$$

for all $\alpha \in \Omega^1$, so that

$$E \simeq E_0 \oplus T^*M,$$

where E_0 denotes the contraction-free part of E .

Definition 2.1. Two torsion-free connections ∇ and ∇' on TM are called *projectively equivalent* if every geodesic of ∇ can be reparametrised to become a geodesic of ∇' . A *projective structure* on M is an equivalence class $\mathfrak{p}$ of projectively equivalent torsion-free connections on TM . A manifold M equipped with a projective structure $\mathfrak{p}$ will be called a *projective manifold*.

Remark 2.2. By a geodesic of ∇ we mean an immersed curve $\gamma : I \rightarrow M$ which satisfies $\nabla_{\dot{\gamma}} \dot{\gamma} = 0$.

We have – see [Spi79] for a modern reference:

Lemma 2.3 (Cartan, Eisenhart, Weyl). *Two torsion-free connections ∇ and ∇' on TM are projectively equivalent if and only if $(\nabla - \nabla')_0 = 0$.*

2.2. A choice of metric

Besides a projective structure p , we now also fix a Riemannian metric g on M . The metric $g = (g_{ij})$ equips each real tensor bundle $E \rightarrow M$ with a bundle metric. Abusing notation, we denote this bundle metric on all the various tensor bundles by $\langle \cdot, \cdot \rangle_g$. For instance, if $\Phi = (\Phi_{jk}^i)$ and $\Psi = (\Psi_{jk}^i)$ are sections of $S^2(T^*M) \otimes TM$, then

$$\langle \Phi, \Psi \rangle_g = \Phi_{jk}^i g^{jb} g^{kc} g_{ia} \Psi_{bc}^a,$$

where we write $g^\sharp = (g^{ij})$ for the dual metric. Correspondingly, we obtain an inner product on the set $\Gamma(E)$ of smooth sections of E by defining

$$\langle\langle s_1, s_2 \rangle\rangle_g = \int_M \langle s_1, s_2 \rangle_g d\mu_g,$$

where $d\mu_g$ denotes the volume form of g and $s_1, s_2 \in \Gamma(E)$. As usual, we write $|s|_g := \sqrt{\langle s, s \rangle_g}$ and likewise $\|s\|_g := \sqrt{\langle\langle s, s \rangle\rangle_g}$ for all $s \in \Gamma(E)$. For what follows, we record the identities

$$(2.2) \quad |\Phi|_{\exp(2f)g}^2 = e^{-2f} |\Phi|_g^2$$

and

$$(2.3) \quad d\mu_{\exp(2f)g} = e^{nf} d\mu_g,$$

where $f \in C^\infty$ and $\Phi \in \Gamma(S^2(T^*M) \otimes TM)$.

The bundle E_0 with $E = S^2(T^*M) \otimes TM$ is irreducible as a $\mathrm{GL}^+(n, \mathbb{R})$ -bundle, it is however not irreducible as a $\mathrm{SO}(n)$ -bundle. Consider the metric trace

$$\mathrm{tr}_g : \Gamma(E_0) \rightarrow \Gamma(TM), \quad \Phi_{jk}^i \mapsto g^{jk} \Phi_{jk}^i$$

and the inclusion

$$\iota_g : \Gamma(TM) \rightarrow \Gamma(E_0), \quad X \mapsto (g \otimes X)_0$$

which satisfy

$$\mathrm{tr}_g \circ \iota_g = \frac{(n+2)(n-1)}{(n+1)} \mathrm{Id}_{\Gamma(TM)}.$$

Consequently, we can decompose $\Phi \in \Gamma(E_0)$ as

$$(2.4) \quad X = \frac{1}{m} \mathrm{tr}_g \Phi \quad \text{and} \quad A = \Phi - (g \otimes X)_0,$$

where $m = \frac{(n+2)(n-1)}{(n+1)}$. By construction, $\mathrm{tr}_g A$ vanishes identically. Furthermore, a simple computation shows that this decomposition is orthogonal. That is, if Z is a vector field on M and Φ a section of E_0 which satisfies $\mathrm{tr}_g \Phi \equiv 0$, then $\langle (g \otimes Z)_0, \Phi \rangle_g$ vanishes identically as well. Moreover, for the decomposition (2.4), we have

$$(2.5) \quad |\Phi|_g^2 = |A|_g^2 + m|X|_g^2,$$

since $|(g \otimes Y)_0|_g^2 = m|Y|_g^2$ for every vector field Y on M .

Remark 2.4. Also observe that for $\Psi \in \Gamma(E_0)$ and a 1-form α on M we have

$$(2.6) \quad \langle \Psi, \text{Sym}(\alpha) \rangle_g = 0,$$

as can be verified with a simple computation.

2.3. A projectively invariant functional

Let $\text{Riem}(M)$ denote the set of Riemannian metrics on M . For a projective structure $\mathfrak{p}$ and orientation on M we consider the following projectively invariant non-negative functional

$$\mathcal{E} : \text{Riem}(M) \rightarrow \mathbb{R}, \quad g \mapsto \text{Vol}_M(g)^{(2-n)/n} \int_M |(\nabla - {}^g\nabla)_0|^2_g d\mu_g,$$

where $\nabla \in \mathfrak{p}$ and ${}^g\nabla$ denotes the Levi-Civita connection of g . Note that the functional attains the value 0 if and only if there exists a Riemannian metric g whose Levi-Civita connection belongs to $\mathfrak{p}$. Also observe that if $\hat{\mathcal{F}} : \text{Riem}(M) \rightarrow \mathbb{R}$ is a functional on the set of Riemannian metrics for which there exists a constant k such that $\hat{\mathcal{F}}(e^{2c}g) = e^{kc}\hat{\mathcal{F}}(g)$ for all constants c and metrics g , then the functional $\mathcal{F}$ defined by $\mathcal{F}(g) = \text{Vol}_M(g)^{-k/n}\hat{\mathcal{F}}(g)$ is invariant under rescaling a metric by a positive constant. Since rescaling a metric by a positive constant does not change its Levi-Civita connection, (2.2) and (2.3) imply that

$$\mathcal{E}(cg) = \mathcal{E}(g)$$

for all positive constants c and all metrics $g \in \text{Riem}(M)$.

Remark 2.5 (Notation). The functional $\mathcal{E}$ depends on a choice of projective structure $\mathfrak{p}$ on M , as do a number of other objects we shall define. To keep the notation uncluttered, we leave this dependency implicit throughout.

3. Variational equations

Let V be a Fréchet space and $U \subset V$ an open subset. The partial derivative of a function $\mathcal{F} : U \rightarrow \mathbb{R}$ at $g \in U$ in the direction of $h \in V$ is defined by

$$\mathcal{F}'_g(h) := \lim_{t \rightarrow 0} \frac{1}{t} (\mathcal{F}(g + th) - \mathcal{F}(g)),$$

provided the limit exists. The set $\Gamma(S^2(T^*M))$ is a Fréchet space and the set of Riemannian metrics on M is an open subset thereof, hence $\text{Riem}(M)$ is a Fréchet manifold whose tangent space at g is canonically isomorphic to $\Gamma(S^2(T^*M))$.

We say a Riemannian metric g on M is *critical* for $\mathcal{E}$ provided $\mathcal{E}'_g(h) = 0$ for all $h \in T_g\text{Riem}(M) \simeq \Gamma(S^2(T^*M))$. Likewise, we say a metric g on M is *conformally critical* for $\mathcal{E}$, provided $\mathcal{E}'_g(h) = 0$ for all h of the form $h = fg$ with $f \in C^\infty$.

For a Riemannian metric g on M , $h \in \Gamma(S^2(T^*M))$ and $t \in \mathbb{R}$, we write $g_t = g + th$ as well as

$$\Phi_{g_t} := (\nabla - {}^{g_t}\nabla)_0 \quad \text{and} \quad \Phi_g := \Phi_{g_0}.$$

Let Γ denote the map which assigns to a Riemannian metric g its Levi-Civita connection and let $\Gamma'_g : \Gamma(S^2(T^*M)) \rightarrow \Gamma(E)$ denote its linearisation at $g \in \text{Riem}(M)$

$$\Gamma'_g(h) = \lim_{t \rightarrow 0} \frac{1}{t} ({}^{g_t}\nabla - {}^g\nabla).$$

Using Γ'_g we obtain a linear first order differential operator

$$\ell_g : \Gamma(S^2(T^*M)) \rightarrow \Gamma(E_0), \quad h \mapsto \left(\Gamma'_g(h) \right)_0.$$

We let

$$\ell_g^* : \Gamma(E_0) \rightarrow \Gamma(S^2(T^*M))$$

denote the formal adjoint satisfying

$$(3.1) \quad \int_M \langle \Psi, \ell_g(h) \rangle_g d\mu_g = \int_M \langle \ell_g^*(\Psi), h \rangle_g d\mu_g$$

for all $h \in \Gamma(S^2(T^*M))$ and all $\Psi \in \Gamma(E_0)$.

Furthermore, for a section Ψ of E_0 we let $T_g(\Psi)$ denote the unique symmetric covariant 2-tensor field so that

$$\left. \frac{\partial}{\partial t} \right|_{t=0} |\Psi|_{g_t}^2 d\mu_{g_t} = \langle T_g(\Psi), h \rangle_g d\mu_g$$

for all $h \in \Gamma(S^2(T^*M))$. In analogy with the physics literature, we call $T_g(\Psi)$ the *stress-energy tensor* of Ψ . The integrand $|\Phi_{g_t}|_{g_t}^2 d\mu_{g_t}$ depends on t through two distinct quantities: the section Φ_{g_t} and the metric pairing together with the volume form $|\cdot|_{g_t}^2 d\mu_{g_t}$. The chain rule therefore gives

$$(3.2) \quad \left. \frac{\partial}{\partial t} \right|_{t=0} |\Phi_{g_t}|_{g_t}^2 d\mu_{g_t} = \left. \frac{\partial}{\partial t} \right|_{t=0} |\Phi_g|_{g_t}^2 d\mu_{g_t} + \left. \frac{\partial}{\partial t} \right|_{t=0} |\Phi_{g_t}|_g^2 d\mu_g,$$

where in the first term on the right the section is frozen at Φ_g , and in the second the pairing and volume are frozen at g . By the definition of the stress-energy tensor, the first term equals $\langle T_g(\Phi_g), h \rangle_g d\mu_g$. By the Leibniz rule applied to the inner product, the second term equals $2\langle \Phi_g, \left. \frac{\partial}{\partial t} \right|_{t=0} \Phi_{g_t} \rangle_g d\mu_g$. We thus obtain

$$(3.3) \quad \left. \frac{\partial}{\partial t} \right|_{t=0} |\Phi_{g_t}|_{g_t}^2 d\mu_{g_t} = \langle T_g(\Phi_g), h \rangle_g d\mu_g + 2\langle \Phi_g, \left. \frac{\partial}{\partial t} \right|_{t=0} \Phi_{g_t} \rangle_g d\mu_g.$$

Writing

$$\Phi_{g_t} = (\nabla - {}^g\nabla - ({}^{g_t}\nabla - {}^g\nabla))_0,$$

we observe that the difference $\nabla - {}^g\nabla$ is independent of t . Setting $H := \Gamma'_g(h)$, the definition of Γ'_g gives

$$\left. \frac{\partial}{\partial t} \right|_{t=0} ({}^{g_t}\nabla - {}^g\nabla) = H,$$

and since $\ell_g(h) = H_0$ by definition of ℓ_g , we conclude

$$\left. \frac{\partial}{\partial t} \right|_{t=0} \Phi_{g_t} = -H_0 = -\ell_g(h).$$

Substituting into (3.3) thus yields

$$(3.4) \quad \left. \frac{\partial}{\partial t} \right|_{t=0} |\Phi_{g_t}|_{g_t}^2 d\mu_{g_t} = \langle T_g(\Phi_g), h \rangle_g d\mu_g - 2\langle \Phi_g, \ell_g(h) \rangle_g d\mu_g.$$

Now consider

$$\hat{\varepsilon}(g) := \int_M |(\nabla - {}^g\nabla)_0|^2 d\mu_g = \int_M |\Phi_g|_g^2 d\mu_g = \|\Phi_g\|_g^2$$

so that, interchanging $\frac{\partial}{\partial t}\big|_{t=0}$ with $\int_M$ and integrating (3.4) over M , the first two equalities below follow; the third uses the defining property (3.1) of ℓ_g^* to push ℓ_g across the inner product:

$$\begin{aligned} \hat{\varepsilon}'_g(h) &= \frac{\partial}{\partial t}\bigg|_{t=0} \int_M |\Phi_{gt}|_{g_t}^2 d\mu_{g_t} = \int_M \langle T_g(\Phi_g), h \rangle_g - 2\langle \Phi_g, \ell_g(h) \rangle_g d\mu_g \\ &= \int_M \langle T_g(\Phi_g) - 2\ell_g^*(\Phi_g), h \rangle_g d\mu_g. \end{aligned}$$

Thinking of the volume of M with respect to a metric as a functional $\text{Vol}_M : \text{Riem}(M) \rightarrow \mathbb{R}$, we have the standard identity

$$(\text{Vol}_M)'_g(h) = \frac{1}{2} \int_M (\text{tr}_g h) d\mu_g = \frac{1}{2} \int_M \langle g, h \rangle_g d\mu_g.$$

Since $\varepsilon(g) = \text{Vol}_M(g)^{(2-n)/n} \hat{\varepsilon}(g)$, the product rule gives

$$\varepsilon'_g(h) = \text{Vol}_M(g)^{(2-n)/n} \hat{\varepsilon}'_g(h) + \frac{2-n}{n} \text{Vol}_M(g)^{(2-n)/n-1} \hat{\varepsilon}(g) (\text{Vol}_M)'_g(h).$$

Substituting the formula for $(\text{Vol}_M)'_g(h)$ above and using $\hat{\varepsilon}(g) = \|\Phi_g\|_g^2$, we obtain

$$\varepsilon'_g(h) = \text{Vol}_M(g)^{(2-n)/n} \left[\hat{\varepsilon}'_g(h) - \left(\frac{n-2}{2n} \right) \frac{\|\Phi_g\|_g^2}{\text{Vol}_M(g)} \int_M \langle g, h \rangle_g d\mu_g \right].$$

Combining this with the expression for $\hat{\varepsilon}'_g(h)$ derived above, and setting $k := \left(\frac{n-2}{2n} \right) \frac{\|\Phi_g\|_g^2}{\text{Vol}_M(g)}$, we conclude

$$(3.5) \quad \varepsilon'_g(h) = \text{Vol}_M(g)^{(2-n)/n} \int_M \langle T_g(\Phi_g) - 2\ell_g^*(\Phi_g) - kg, h \rangle_g d\mu_g.$$

By the fundamental lemma of the calculus of variations, $\varepsilon'_g(h) = 0$ for all $h \in \Gamma(S^2(T^*M))$ if and only if the symmetric tensor $T_g(\Phi_g) - 2\ell_g^*(\Phi_g) - kg$ vanishes identically. We thus have:

Proposition 3.1. *A Riemannian metric g is critical for ε if and only if*

$$T_g(\Phi_g) - 2\ell_g^*(\Phi_g) = kg,$$

where the constant k is given by $k = \left(\frac{n-2}{2n} \right) \frac{\|\Phi_g\|_g^2}{\text{Vol}_M(g)}$.

We also need:

Lemma 3.2. *For $\Psi \in \Gamma(E_0)$, the stress-energy tensor $T_g(\Psi) \in \Gamma(S^2(T^*M))$ is given by*

$$T_g(\Psi) = \frac{1}{2} |\Psi|_g^2 g + \Psi \otimes_g \Psi,$$

where

$$(\Psi \otimes_g \Psi)_{ij} = \Psi_{uv}^k \Psi_{bc}^a g_{ia} g_{jk} g^{ub} g^{vc} - 2\Psi_{jc}^k \Psi_{ib}^a g_{ak} g^{bc}.$$

In particular

$$(3.6) \quad \text{tr}_g(T_g(\Psi)) = \left(\frac{n}{2} - 1 \right) |\Psi|_g^2.$$

Proof. Recall the standard identities

$$\left. \frac{\partial}{\partial t} \right|_{t=0} d\mu_{g_t} = \frac{1}{2}(\text{tr}_g h) d\mu_g \quad \text{and} \quad \left. \frac{\partial}{\partial t} \right|_{t=0} (g_t^\sharp)^{ij} = -h^{ij}$$

where we raise indices of h with the metric g . Then

$$\begin{aligned} |\Psi|_{g_t}^2 d\mu_{g_t} &= \Psi_{ij}^k \Psi_{bc}^a (g_{ak} + th_{ak}) (g^{ib} - th^{ib} + \mathcal{O}(t^2)) \\ &\quad (g^{jc} - th^{jc} + \mathcal{O}(t^2)) \left(1 + \frac{1}{2} t g_{pq} h^{pq} + \mathcal{O}(t^2) \right) d\mu_g. \end{aligned}$$

From this we calculate

$$\begin{aligned} \left. \frac{\partial}{\partial t} \right|_{t=0} |\Psi|_{g_t}^2 d\mu_{g_t} &= \left[\Psi_{uv}^k \Psi_{bc}^a g_{ia} g_{jk} g^{ub} g^{vc} - 2 \Psi_{jc}^k \Psi_{ib}^a g_{ak} g^{bc} \right. \\ &\quad \left. + \frac{1}{2} |\Psi|_g^2 g_{ij} \right] h^{ij} d\mu_g, \end{aligned}$$

so that

$$\left. \frac{\partial}{\partial t} \right|_{t=0} |\Psi|_{g_t}^2 d\mu_{g_t} = \langle T_g(\Psi), h \rangle_g d\mu_g,$$

as claimed. The identity (3.6) follows from an elementary computation which we omit. $\square$

Lemma 3.3. *For $g \in \text{Riem}(M)$ the operator*

$$\ell_g^* : \Gamma(E_0) \rightarrow \Gamma(S^2(T^*M))$$

is given by

$$\Psi_{jk}^i \mapsto \frac{1}{2} \left(g \nabla_k \Psi_{ij}^k - g^{uv} g_{ik} g \nabla_u \Psi_{jv}^k - g^{uv} g_{jk} g \nabla_u \Psi_{iv}^k \right).$$

Proof. We need to show that for $g \in \text{Riem}(M)$, $h \in \Gamma(S^2(T^*M))$ and $\Psi \in \Gamma(E_0)$, we have

$$\int_M \langle \Psi, \ell_g(h) \rangle_g d\mu_g = \int_M \langle \ell_g^*(\Psi), h \rangle_g d\mu_g.$$

With our notation $H = \Gamma'_g(h)$, an elementary computation gives

$$H_{ij}^k = \frac{1}{2} g^{kl} (g \nabla_i h_{jl} + g \nabla_j h_{il} - g \nabla_l h_{ij}).$$

Recall from (2.1) that

$$H = H_0 + \frac{1}{(n+1)} \text{Sym}(\text{Con}(H)).$$

Since $\Psi \in \Gamma(E_0)$ and $\ell_g(h) = H_0$, we have

$$\langle \Psi, H \rangle_g = \langle \Psi, H_0 \rangle_g + \frac{1}{(n+1)} \langle \Psi, \text{Sym}(\text{Con}(H)) \rangle_g = \langle \Psi, H_0 \rangle_g = \langle \Psi, \ell_g(h) \rangle_g,$$

where we use (2.6). Contracting $g_{ka} g^{al} = \delta_k^l$ gives

$$\langle \Psi, H \rangle_g = \frac{1}{2} \Psi_{ij}^k g^{ib} g^{jc} (g \nabla_b h_{ck} + g \nabla_c h_{bk} - g \nabla_k h_{bc}).$$

The first two terms contribute equally (swap $i \leftrightarrow j$, $b \leftrightarrow c$ and use $\Psi_{ij}^k = \Psi_{ji}^k$), so that

$$\langle \Psi, H \rangle_g = \Psi_{ij}^k g^{ib} g^{jcg} \nabla_b h_{ck} - \frac{1}{2} \Psi_{ij}^k g^{ib} g^{jcg} \nabla_k h_{bc}.$$

Consider the vector field $Y = (Y^l)$ on M with

$$Y^l = g^{il} g^{jc} \Psi_{ij}^k h_{ck}.$$

Since $\int_M \operatorname{div}_g Y \, d\mu_g = 0$ by Stokes' theorem, we obtain

$$\int_M g^{ib} g^{jc} \nabla_b (\Psi_{ij}^k h_{ck}) d\mu_g = 0$$

and hence

$$\int_M g^{ib} g^{jc} \Psi_{ij}^k g \nabla_b h_{ck} d\mu_g = - \int_M g^{ib} g^{jc} (g \nabla_b \Psi_{ij}^k) h_{ck} d\mu_g.$$

Like-wise we obtain

$$-\frac{1}{2} \int_M \Psi_{ij}^k g^{ib} g^{jcg} \nabla_k h_{bc} d\mu_g = \frac{1}{2} \int_M g^{ib} g^{jc} (g \nabla_k \Psi_{ij}^k) h_{bc} d\mu_g.$$

In the second integral, substituting $h_{bc} = g_{bp} g_{cq} h^{pq}$ gives $\frac{1}{2} (g \nabla_k \Psi_{pq}^k) h^{pq}$. In the first, substituting $h_{ck} = g_{cp} g_{kq} h^{pq}$ gives $-g^{ub} g_{kq} (g \nabla_b \Psi_{up}^k) h^{pq}$. Symmetrising in p, q and relabelling summation indices (using $\Psi_{up}^k = \Psi_{pu}^k$) yields

$$[\ell_g^*(\Psi)]_{pq} = \frac{1}{2} \left(g \nabla_k \Psi_{pq}^k - g^{uv} g_{pk} g \nabla_u \Psi_{qv}^k - g^{uv} g_{qk} g \nabla_u \Psi_{pv}^k \right). \quad \square$$

3.1. Decomposing Φ_g

Decomposing Φ_g as in (2.4)

$$X_g := \frac{1}{m} \operatorname{tr}_g \Phi_g \quad \text{and} \quad A_g := \Phi_g - (g \otimes X_g)_0$$

it turns out that A_g depends only on the conformal equivalence class $[g]$ of g . To this end let $f \in C^\infty$ and consider $\tilde{g} = \exp(2f)g$. Using the identity (see [Bes87, Thm. 1.159])

$$(3.7) \quad \tilde{g} \nabla = g \nabla - g \otimes \operatorname{grad}_g f + \operatorname{Sym}(\operatorname{d}f),$$

we obtain

$$\Phi_{\tilde{g}} = (\nabla - \tilde{g} \nabla)_0 = \Phi_g + (g \otimes \operatorname{grad}_g f)_0 = \Phi_g + g \otimes \operatorname{grad}_g f - \frac{1}{(n+1)} \operatorname{Sym}(\operatorname{d}f).$$

where we use that $\operatorname{Con}(g \otimes \operatorname{grad}_g f) = \operatorname{d}f$. Hence we compute

$$(3.8) \quad \begin{aligned} X_{\tilde{g}} &= \frac{1}{m} \operatorname{tr}_{\tilde{g}} \Phi_{\tilde{g}} = e^{-2f} \left(X_g + \frac{n}{m} \operatorname{grad}_g f - \frac{2}{(n+1)m} \operatorname{grad}_g f \right) \\ &= e^{-2f} (X_g + \operatorname{grad}_g f). \end{aligned}$$

This also gives

$$\begin{aligned}
A_{\tilde{g}} &= \Phi_{\tilde{g}} - (\tilde{g} \otimes X_{\tilde{g}})_0 \\
&= \Phi_g + g \otimes \text{grad}_g f - \frac{1}{(n+1)} \text{Sym}(\text{d}f) - \left(g \otimes (X_g + \text{grad}_g f) \right)_0 \\
&= A_g + g \otimes \text{grad}_g f - \frac{1}{(n+1)} \text{Sym}(\text{d}f) - (g \otimes \text{grad}_g f)_0 \\
&= A_g - \frac{1}{(n+1)} \text{Sym}(\text{d}f) + \frac{1}{(n+1)} \text{Sym}(\text{Con}(g \otimes \text{grad}_g f)) = A_g.
\end{aligned}$$

We will henceforth write $A_{[g]}$ instead of A_g . The conformal invariance of $A_{[g]}$ was previously observed in [Met20, Thm. 2.4]. We included the argument for the convenience of the reader.

For what follows we recall the standard identity for a vector field Y on M

$$\text{div}_{\tilde{g}} Y = \text{div}_g Y + nY(f),$$

which, together with (3.8), gives

$$(3.9) \quad \text{div}_{\tilde{g}} X_{\tilde{g}} = e^{-2f} \left(\text{div}_g X_g + (n-2)X_g(f) - \Delta_g f + (n-2)|\text{d}f|_g^2 \right),$$

where $\Delta_g = -\text{div}_g \text{grad}_g$.

3.2. Conformal variations

As an immediate corollary of Proposition 3.1 we have:

Corollary 3.4. *A metric g is conformally critical for $\mathcal{E}$ if and only if*

$$(3.10) \quad \text{div}_g(\text{tr}_g \Phi_g) = \left(\frac{n}{2} - 1 \right) \left(|\Phi_g|_g^2 - \frac{\|\Phi_g\|_g^2}{\text{Vol}_M(g)} \right).$$

Proof. Observe that we have

$$\text{tr}_g \left(\ell_g^*(\Phi_g) \right) = \frac{1}{2} \text{div}_g(\text{tr}_g \Phi_g),$$

where

$$\text{div}_g(\text{tr}_g \Phi_g) = {}^g \nabla_i g^{jk} (\Phi_g)^i_{jk}.$$

Furthermore, recall (3.6)

$$\text{tr}_g(T_g(\Phi_g)) = \left(\frac{n}{2} - 1 \right) |\Phi_g|_g^2.$$

For $h = fg$ with $f \in C^\infty$, we obtain

$$\begin{aligned}
\mathcal{E}'_g(fg) &= \text{Vol}_M(g)^{(2-n)/n} \int_M f \text{tr}_g \left(T_g(\Phi_g) - 2\ell_g^*(\Phi_g) \right) - kn f d\mu_g \\
&= \text{Vol}_M(g)^{(2-n)/n} \int_M f \left[\left(\frac{n}{2} - 1 \right) |\Phi_g|_g^2 - \text{div}_g(\text{tr}_g \Phi_g) - kn \right] d\mu_g,
\end{aligned}$$

and the claim follows. $\square$

4. The surface case

4.1. Example: Projective structures from holomorphic cubic differentials

Let (M, g) be an oriented Riemannian 2-manifold and let $J : TM \rightarrow TM$ denote counter clockwise rotation by $\pi/2$ with respect to the orientation and metric g . We consider a cubic differential C on the Riemann surface (M, J) . Its real part is a totally symmetric $(0, 3)$ tensor field $A = (A_{ijk})$ on M that is totally trace-free with respect to g , that is, $g^{ij}A_{ijk}$ vanishes identically. Define $\alpha = (\alpha_{jk}^i) \in \Gamma(E)$ by $\alpha_{jk}^i = g^{il}A_{ljk}$ and consider the projective structure $\mathfrak{p}$ arising from ${}^g\nabla + \alpha$. By construction, $\alpha \in \Gamma(E_0)$ and $\Phi_g = \alpha$. Applying Proposition 3.1, the variational equations become $T_g(\alpha) = 2\ell_g^*(\alpha)$. By (3.6), the tensor field $T_g(\alpha)$ is totally trace-free with respect to g and hence encodes a quadratic differential on (M, J) . The map T_g thus gives rise to a quadratic form on $\Gamma(K_{M,J}^3)$ with values in $\Gamma(K_{M,J}^2)$, where $K_{M,J}$ denotes the canonical bundle of (M, J) . Unsurprisingly, this quadratic form is trivial (as can be checked with a direct computation), hence $T_g(\alpha)$ vanishes identically and the variational equations simplify to $\ell_g^*(\alpha) = 0$. A computation gives $2\ell_g^*(\alpha) = -\operatorname{div}_g \alpha$ with $(\operatorname{div}_g \alpha)_{ij} = {}^g\nabla_k \alpha_{ij}^k$. This last condition is well known to be equivalent to the holomorphicity of C . From this we obtain:

Theorem A. *Let $(M, \mathfrak{p})$ be a closed oriented properly convex projective surface of negative Euler characteristic. Then the Blaschke metric of $\mathfrak{p}$ is a critical point for $\mathcal{E}$.*

Proof. By [Lof01, Lab07], on a closed oriented surface M of negative Euler characteristic every *properly convex projective structure* $\mathfrak{p}$ arises from a unique pair (g, C) with C holomorphic via the above construction. That is, the torsion-free connection ${}^g\nabla + \alpha$ on TM is a representative connection of $\mathfrak{p}$. Additionally, the pair satisfies $K_g = -1 + 2|C|_g^2$, where K_g denotes the Gauss curvature of g . The metric g is known as the *Blaschke metric* of $\mathfrak{p}$. The above computations then show that $T_g(\alpha) = 0$ and $\ell_g^*(\alpha) = 0$ by the holomorphicity of C , so the Blaschke metric g is critical for $\mathcal{E}$. $\square$

4.2. Existence and uniqueness of conformally critical metrics

Corollary 3.4 implies that for $n = 2$ a Riemannian metric g is conformally critical if and only if $\operatorname{div}_g X_g = 0$. Using (3.9), looking for a conformally critical metric $\tilde{g} = e^{2f}g$ in the conformal equivalence class of g thus gives the PDE

$$\operatorname{div}_{\tilde{g}} X_{\tilde{g}} = e^{-2f} (\operatorname{div}_g X_g - \Delta_g f) = 0.$$

Hence we obtain:

Theorem B. *Let $(M, \mathfrak{p})$ be a closed oriented projective surface. Then every conformal structure on M contains a conformally critical metric, unique up to homothety.*

Proof. Recall that for $u \in C^\infty$ the Poisson equation $\Delta_g f = u$ on a closed oriented Riemannian manifold (M, g) has a smooth solution, unique up to adding a constant, provided $\int_M u d\mu_g = 0$. In our case $u = \operatorname{div}_g X_g$, hence the claim is a consequence of Stokes' theorem. $\square$

5. The projective Yamabe equation

5.1. The projective-conformal Laplacian

From now on we assume $n \geq 3$. Consider a closed oriented conformal n -manifold $(M, [g])$. Recall that the Yamabe problem asks whether there exists a metric $\tilde{g} \in [g]$ whose scalar curvature $S_{\tilde{g}}$ is constant. For $f \in C^\infty$ the scalar curvature of the metric $\tilde{g} = \exp(2f)g$ is

$$S_{\tilde{g}} = e^{-2f} \left(S_g + 2(n-1)\Delta_g f - (n-1)(n-2)|df|_g^2 \right),$$

where $\Delta_g f = -\operatorname{div}_g \operatorname{grad}_g f$. Writing $\tilde{g} = u^{4/(n-2)}g$ for a positive function u , the previous equation becomes

$$4 \left(\frac{n-1}{n-2} \right) \Delta_g u + S_g u = u^{(n+2)/(n-2)} S_{\tilde{g}}.$$

Finding a metric $\tilde{g} \in [g]$ with constant scalar curvature C thus amounts to solving $L_g u = C u^{(n+2)/(n-2)}$, where $L_g = 4 \left(\frac{n-1}{n-2} \right) \Delta_g + S_g$ denotes the so-called *conformal Laplacian*.

We now return to the problem of finding a conformally critical metric on a closed oriented projective manifold of dimension $n \geq 3$. Writing

$$(5.1) \quad \mathcal{S}_g := \frac{2(n+1)}{(n+2)} \left[\left(\frac{n}{2} - 1 \right) |\Phi_g|_g^2 - \operatorname{div}_g(\operatorname{tr}_g \Phi_g) \right]$$

the equation (3.10) is equivalent to the statement that a metric g is conformally critical for $\mathcal{E}$ if and only if $\mathcal{S}_g$ is constant.

Lemma 5.1. *Let $f \in C^\infty$. Under a conformal change $g \mapsto \tilde{g} = e^{2f}g$, the function $\mathcal{S}_g$ transforms as the scalar curvature, that is,*

$$\mathcal{S}_{\tilde{g}} = e^{-2f} (\mathcal{S}_g + 2(n-1)\Delta_g f - (n-1)(n-2)|df|_g^2),$$

where $\Delta_g f = -\operatorname{div}_g \operatorname{grad}_g f$.

Proof. From the formula

$$\Phi_{\tilde{g}} = \Phi_g + g \otimes \operatorname{grad}_g f - \frac{1}{(n+1)} \operatorname{Sym}(df)$$

we compute

$$\begin{aligned}
 |\Phi_{\tilde{g}}|_{\tilde{g}}^2 &= e^{-2f} |\Phi_g + (g \otimes \text{grad}_g f)_0|_g^2 \\
 (5.2) \quad &= e^{-2f} \left(|\Phi_g|_g^2 + 2mX_g(f) + |(g \otimes \text{grad}_g f)_0|_g^2 \right) \\
 &= e^{-2f} \left(|\Phi_g|_g^2 + 2mX_g(f) + m|\text{d}f|_g^2 \right),
 \end{aligned}$$

where we have used the identities

$$|\text{Sym}(\text{d}f)|_g^2 = 2(n+1)|\text{d}f|_g^2 \quad \text{and} \quad |g \otimes \text{grad}_g f|_g^2 = n|\text{d}f|_g^2$$

as well as

$$\langle g \otimes \text{grad}_g f, \text{Sym}(\text{d}f) \rangle_g = 2|\text{d}f|_g^2.$$

Now using (3.9) and (5.2) we obtain

$$\begin{aligned}
 \mathcal{S}_{\tilde{g}} &= \frac{(n-1)(n-2)}{m} |\Phi_{\tilde{g}}|_{\tilde{g}}^2 - 2(n-1) \text{div}_{\tilde{g}} X_{\tilde{g}} \\
 &= e^{-2f} \left[\mathcal{S}_g + 2(n-1)(n-2)X_g(f) + (n-1)(n-2)|\text{d}f|_g^2 \right. \\
 &\quad \left. - 2(n-1)(n-2)X_g(f) + 2(n-1)\Delta_g f - 2(n-1)(n-2)|\text{d}f|_g^2 \right] \\
 &= e^{-2f} \left(\mathcal{S}_g + 2(n-1)\Delta_g f - (n-1)(n-2)|\text{d}f|_g^2 \right),
 \end{aligned}$$

as claimed. $\square$

Finding a metric $\tilde{g}$ in a conformal equivalence class $[g]$ which is conformally critical for $\mathcal{E}$ is thus equivalent to solving an equation of Yamabe type $\mathcal{L}_g u = C u^{(n+2)/(n-2)}$, where

$$\mathcal{L}_g := 4 \left(\frac{n-1}{n-2} \right) \Delta_g + \mathcal{S}_g.$$

Remark 5.2. We will refer to $\mathcal{L}_g$ as the *projective-conformal Laplacian*.

5.2. The projective scalar curvature

As an immediate consequence of Stokes' theorem and definition (5.1) we have

Proposition 5.3. *Let g be a Riemannian metric on the closed oriented projective manifold $(M, \mathfrak{p})$ of dimension $n \geq 3$. Then*

$$(5.3) \quad \int_M \mathcal{S}_g d\mu_g = \hat{m} \|\Phi_g\|_g^2 \geq 0,$$

where $\hat{m} = \frac{(n+1)(n-2)}{(n+2)}$, with equality if and only if $\mathfrak{p}$ contains the Levi-Civita connection of g .

Denoting by $\sigma(\mathcal{L}_g)$ the spectrum of $\mathcal{L}_g$, we also conclude:

Proposition 5.4. *Let $(M, \mathfrak{p})$ be a closed oriented projective manifold of dimension $n \geq 3$ and g a Riemannian metric on M . Then $\sigma(\mathcal{L}_g)$ is non-negative and moreover, contains 0 if and only if $\mathfrak{p}$ is metrisable by a metric $\tilde{g}$ in the conformal equivalence class of g .*

Proof. For every positive function u on M , writing $\tilde{g} = u^{4/(n-2)}g$ and using $\mathcal{S}_{\tilde{g}} = u^{-(n+2)/(n-2)}\mathcal{L}_g u$ as well as $d\mu_{\tilde{g}} = u^{2n/(n-2)}d\mu_g$, we obtain

$$\int_M u \mathcal{L}_g u d\mu_g = \int_M u^{2n/(n-2)} \mathcal{S}_{\tilde{g}} d\mu_g = \int_M \mathcal{S}_{\tilde{g}} d\mu_{\tilde{g}} = \hat{m} \|\Phi_{\tilde{g}}\|_{\tilde{g}}^2 \geq 0.$$

Since, by standard elliptic PDE theory, the first eigenfunction of the self-adjoint elliptic operator $\mathcal{L}_g$ can be chosen positive, it follows that the first eigenvalue λ_1 of $\mathcal{L}_g$ satisfies $\lambda_1 \geq 0$ and hence the full spectrum is non-negative. If $\lambda_1 = 0$, the corresponding positive eigenfunction u satisfies $\mathcal{L}_g u = 0$, so that $\|\Phi_{\tilde{g}}\|_{\tilde{g}}^2 = 0$ for $\tilde{g} = u^{4/(n-2)}g$, meaning $\mathfrak{p}$ contains the Levi-Civita connection of $\tilde{g} \in [g]$. Conversely, if $\tilde{g} = u^{4/(n-2)}g$ metrises $\mathfrak{p}$, then $\Phi_{\tilde{g}} = 0$ and hence $\mathcal{L}_g u = u^{(n+2)/(n-2)}\mathcal{S}_{\tilde{g}} = 0$, so that 0 is in the spectrum of $\mathcal{L}_g$. $\square$

Because of the analogy to the scalar curvature, we will refer to $\mathcal{S}_g$ as the *projective scalar curvature* of g . For what follows it is advantageous to express $\mathcal{S}_g$ in terms of the usual scalar curvature of g , the “scalar curvature” of a certain Weyl connection and $A_{[g]}$. It follows from (3.7) and (3.8) that the torsion-free connection

$$(5.4) \quad [g]\nabla := {}^g\nabla + g \otimes X_g - \text{Sym}(X_g^{\flat})$$

depends only on the conformal equivalence class of g . Here $\flat$ is taken with respect to g . The connection $[g]\nabla$ on TM is a *Weyl connection* for $[g]$, that is, it is torsion-free and its parallel transport maps are angle preserving with respect to $[g]$. In fact, all Weyl connections for $[g]$ are of the form (5.4) for some metric $g \in [g]$ and some vector field X . For a pair (g, X) we write

$$S_{g,X} := \text{tr}_g \text{Ric} \left([g]\nabla \right),$$

where $\text{Ric} \left([g]\nabla \right)$ denotes the Ricci curvature of the Weyl connection $[g]\nabla$ determined by (g, X) . A standard calculation – see Section A – gives:

$$(5.5) \quad S_{g,X} = S_g + 2(n-1) \text{div}_g X - (n-2)(n-1)|X|_g^2,$$

where S_g denotes the scalar curvature of g .

Proposition 5.5. *For the projective scalar curvature we have*

$$\mathcal{S}_g = S_g - S_{g,X_g} + \hat{m}|A_{[g]}|_g^2.$$

Proof. Using the definition (5.1), (5.5) and (2.5), we compute

$$\begin{aligned} \mathcal{S}_g &= \frac{2(n+1)}{(n+2)} \left[\left(\frac{n}{2} - 1 \right) |\Phi_g|_g^2 - \text{div}_g(\text{tr}_g \Phi_g) \right] \\ &= \hat{m}|\Phi_g|_g^2 - 2(n-1) \text{div}_g(X_g) \\ &= \hat{m}|A_{[g]}|_g^2 + \hat{m}m|X_g|_g^2 - 2(n-1) \text{div}_g(X_g) \\ &= S_g - (S_g + 2(n-1) \text{div}_g(X_g) - (n-2)(n-1)|X_g|_g^2) + \hat{m}|A_{[g]}|_g^2 \\ &= S_g - S_{g,X_g} + \hat{m}|A_{[g]}|_g^2, \end{aligned}$$

where we also used that $\hat{m}m = (n-2)(n-1)$. □

Remark 5.6. In terms of X_g and $A_{[g]}$ we thus have

$$\mathcal{S}_g = \hat{m}|A_{[g]}|_g^2 + (n-2)(n-1)|X_g|_g^2 - 2(n-1)\operatorname{div}_g(X_g).$$

6. Existence of conformally critical metrics

For a Riemannian metric g on the projective manifold $(M, \mathfrak{p})$ we define:

Definition 6.1. The *conformal defect* of g on $(M, \mathfrak{p})$ is the difference of the projective – and standard scalar curvature:

$$V_g := \mathcal{S}_g - S_g = \hat{m}|A_{[g]}|_g^2 - S_{g, X_g}.$$

Remark 6.2.

- (i) Notice that $\mathcal{L}_g = L_g + V_g$, so that V_g measures the defect of the projective-conformal Laplacian to agree with the conformal Laplacian.
- (ii) Recall that we define the conformal locus of g as

$$\Sigma_{[g]} = \{p \in M \mid V_g(p) = 0\}.$$

Under a conformal change $\tilde{g} = e^{2f}g$, the conformal defect satisfies $V_{\tilde{g}} = e^{-2f}V_g$. In particular, the conformal locus does only depend on the conformal equivalence class of g .

6.1. Reduction to the Yamabe problem

In the special case where the conformal locus of $[g]$ is all of M , the projective-conformal Laplacian and the conformal Laplacian agree, hence finding a conformally critical metric for $\mathcal{E}$ amounts to solving the Yamabe problem. The affirmative solution to the Yamabe problem [Yam60] by Trudinger [Tru68], Aubin [Aub76], and Schoen [Sch84] thus implies (we also refer to [LP87] and [DH05] for surveys):

Theorem 6.3. *Let $(M, \mathfrak{p})$ be a closed oriented projective manifold of dimension $n \geq 3$. Suppose the conformal structure $[g]$ on M satisfies $\Sigma_{[g]} = M$. Then $[g]$ contains a conformally critical metric for $\mathcal{E}$.*

Proof. By assumption $V_g \equiv 0$, so $\mathcal{L}_g = L_g$ and the criticality equation reduces to the Yamabe equation, which has a positive smooth solution by the work of Yamabe, Trudinger, Aubin and Schoen. □

6.2. Negative conformal defect

We treat in this subsection the case where the conformal defect V_g takes a negative value at some point of M . Since $V_{\tilde{g}} = e^{-2f}V_g$ for $\tilde{g} = e^{2f}g$, the hypothesis that V_g is negative at a given point is independent of the choice of representative metric in $[g]$.

Theorem 6.4. *Let (M, p) be a closed oriented projective manifold of dimension $n \geq 4$ and $[g]$ a conformal structure on M . Suppose there exists $p_0 \in M$ such that $V_{\tilde{g}}(p_0) < 0$ for some (and hence any) $\tilde{g} \in [g]$. Then $[g]$ contains a conformally critical metric for $\mathcal{E}$.*

Proof. By Section 5.1, finding a metric $\tilde{g} \in [g]$ which is conformally critical for $\mathcal{E}$ is equivalent to finding a smooth positive function u on M that solves $\mathcal{L}_g u = C u^{(n+2)/(n-2)}$ for some constant $C \geq 0$. Setting $h := \frac{n-2}{4(n-1)} \mathcal{S}_g$ and $\lambda := \frac{n-2}{4(n-1)} C$, this rewrites as the elliptic equation of Yamabe type

$$\Delta_g u + h u = \lambda u^{(n+2)/(n-2)}$$

in the sense of [DH05]. The hypothesis $V_g(p_0) < 0$ becomes

$$h(p_0) < \frac{n-2}{4(n-1)} \mathcal{S}_g(p_0),$$

that is, the linear term of the equation drops below the linear term of the standard Yamabe equation at the point p_0 . Existence of a smooth positive solution then follows from [DH05, Thm. 4.1] and the discussion thereafter. $\square$

Combining Theorem 6.3 and Theorem 6.4 we have:

Theorem C. *Let (M, p) be a closed oriented projective manifold of dimension $n \geq 3$ and $[g]$ a conformal structure on M . Suppose that either $\Sigma_{[g]} = M$, or that $n \geq 4$ and there exists $p_0 \in M$ with $V_{\tilde{g}}(p_0) < 0$ for some (and hence any) $\tilde{g} \in [g]$. Then $[g]$ contains a conformally critical metric for $\mathcal{E}$.*

Remark 6.5. Theorem 6.4 covers, in particular, the empty-locus subcase $V_g < 0$ on M . The complementary empty-locus subcase $V_g > 0$ on M remains open.

Appendix A. Ricci curvature of a Weyl connection

Since the formula for the Ricci curvature of a Weyl connection does not seem to be easy to locate in the literature, we include the calculation.

Let ∇ be a torsion-free connection on TM with curvature tensor $R = (R^i_{jkl})$ and let $\Phi = (\Phi^i_{jk})$ be a 1-form on M with values in the endomorphisms of TM satisfying $\Phi(v)(w) = \Phi(w)(v)$ for all $v, w \in T_p M$ and all $p \in M$. Then the torsion-free connection $\hat{\nabla} = \nabla + \Phi$ has curvature

$$\hat{R}^i_{jkl} = R^i_{jkl} + \nabla_k \Phi^i_{lj} - \nabla_l \Phi^i_{kj} + \Phi^u_{lj} \Phi^i_{ku} - \Phi^u_{kj} \Phi^i_{lu}$$

as follows from an elementary computation. In particular, for the Ricci curvature $\hat{R}_{jk} = \hat{R}^l_{jlk}$ of $\hat{\nabla}$ we obtain

$$\hat{R}_{jk} = R_{jk} + \nabla_l \Phi^l_{jk} - \nabla_k \Phi^l_{jl} + \Phi^u_{jk} \Phi^l_{ul} - \Phi^u_{jl} \Phi^l_{ku}.$$

For (g, X) consider the Weyl connection

$$\hat{\nabla} := {}^g\nabla + g \otimes X - \text{Sym}(X^\flat).$$

Take $\Phi = g \otimes X - \text{Sym}(X^\flat)$ and $\nabla = {}^g\nabla$ so that

$$\Phi_{jk}^i = X^i g_{jk} - \delta_j^i \xi_k - \delta_k^i \xi_j,$$

where $\xi_i = g_{ij} X^j$. Then we have

$$\nabla_l \Phi_{jk}^l = \nabla_l (X^l g_{jk} - \delta_j^l \xi_k - \delta_k^l \xi_j) = \text{div}_g(X) g_{jk} - \nabla_j \xi_k - \nabla_k \xi_j$$

Likewise, we obtain

$$\nabla_k \Phi_{jl}^l = -n \nabla_k \xi_j$$

as well as

$$\Phi_{jk}^u \Phi_{ul}^l = 2n \xi_j \xi_k - n |X|_g^2 g_{jk}$$

and

$$\Phi_{jl}^u \Phi_{ku}^l = (n+2) \xi_j \xi_k - 2 |X|_g^2 g_{jk}.$$

From this we compute for the Ricci curvature $\hat{R}_{jk}$ of the Weyl connection $\tilde{\nabla}$

$$\begin{aligned} \hat{R}_{jk} &= R_{jk} + \text{div}_g(X) g_{jk} - \nabla_j \xi_k - \nabla_k \xi_j + n \nabla_k \xi_j \\ &\quad + 2n \xi_j \xi_k - n |X|_g^2 g_{jk} - (n+2) \xi_j \xi_k + 2 |X|_g^2 g_{jk} \\ &= R_{jk} + (n-2) (\nabla_{(j} \xi_{k)} + \xi_j \xi_k - |X|_g^2 g_{jk}) + \text{div}_g(X) g_{jk} - \frac{n}{2} (d\xi)_{jk}, \end{aligned}$$

where R_{jk} denotes the Ricci curvature of ∇ , round brackets on indices denote symmetrisation and square brackets denote anti-symmetrisation. In particular, the symmetric and antisymmetric parts of the Ricci curvature are

$$\hat{R}_{(jk)} = R_{jk} + (n-2) (\nabla_{(j} \xi_{k)} + \xi_j \xi_k - |X|_g^2 g_{jk}) + \text{div}_g(X) g_{jk}$$

and

$$\hat{R}_{[jk]} = -\frac{n}{2} (d\xi)_{jk}.$$

From this we compute

$$S_{g,X} = S_g + 2(n-1) \text{div}_g(X) - (n-2)(n-1) |X|_g^2,$$

where $S_{g,X} := \text{tr}_g \text{Ric}(\hat{\nabla})$ and S_g denotes the scalar curvature of g .

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