



# Metrisability of projective surfaces and pseudo-holomorphic curves

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**ABSTRACT.** We show that the metrisability of an oriented projective surface is equivalent to the existence of pseudo-holomorphic curves. A projective structure  $\mathfrak{p}$  and a volume form  $\sigma$  on an oriented surface  $M$  equip the total space of a certain disk bundle  $Z \rightarrow M$  with a pair  $(J_{\mathfrak{p}}, \mathfrak{J}_{\mathfrak{p}, \sigma})$  of almost complex structures. A conformal structure on  $M$  corresponds to a section of  $Z \rightarrow M$  and  $\mathfrak{p}$  is metrisable by the metric  $g$  if and only if  $[g] : M \rightarrow Z$  is a pseudo-holomorphic curve with respect to  $J_{\mathfrak{p}}$  and  $\mathfrak{J}_{\mathfrak{p}, dA_g}$ .

## 1. Introduction

A *projective structure* on a smooth manifold consists of an equivalence class  $\mathfrak{p}$  of torsion-free connections on its tangent bundle, where two such connections are called equivalent if they have the same geodesics up to parametrisation. A projective structure  $\mathfrak{p}$  is called *metrisable* if it contains the Levi-Civita connection of some Riemannian metric. The problem of (locally) characterising the projective structures that are metrisable was first studied in the work of R. Liouville [17] in 1889, but was solved only relatively recently by Bryant, Dunajski and Eastwood for the case of two dimensions [2]. Since then, there has been renewed interest in the problem, see [5, 6, 8, 10, 11, 13, 14, 25, 27] for related recent work.

The purpose of this short note is to show that in the case of an oriented projective surface  $(M, \mathfrak{p})$ , the metrisability of  $\mathfrak{p}$  is equivalent to the existence of certain pseudo-holomorphic curves.

An orientation compatible complex structure on  $M$  corresponds to a section of the bundle  $\pi : Z \rightarrow M$  whose fibre at  $x \in M$  consists of the orientation compatible linear complex structures on  $T_x M$ . The choice of a torsion-free connection  $\nabla$  on  $TM$  equips  $Z$  with an almost complex structure  $J$  [7, 26]. Namely, at  $j \in Z$  we lift  $j$  horizontally and take a natural complex structure on each fibre vertically. It turns out that  $J$  is always integrable and does only depend on the projective equivalence class  $\mathfrak{p}$  of  $\nabla$ , we thus denote it by  $J_{\mathfrak{p}}$ . Reversing the orientation on each fibre yields another almost complex structure  $\mathfrak{J}$  which is however never integrable and is not projectively invariant. Fixing a volume form  $\sigma$  on the projective surface  $(M, \mathfrak{p})$  determines a unique representative connection  ${}^{\sigma}\nabla \in \mathfrak{p}$  which preserves  $\sigma$ . We will write  $\mathfrak{J}_{\mathfrak{p}, \sigma}$  for the non-integrable almost complex structure arising from  ${}^{\sigma}\nabla \in \mathfrak{p}$ .

The choice of a conformal structure  $[g]$  on an oriented surface  $M$  defines an orientation compatible complex structure by rotating a tangent vector counter-clockwise by  $\pi/2$  with respect to  $[g]$ . Thus, we may think of a conformal structure as a section  $[g] : M \rightarrow Z$ . Denoting the area form of a Riemannian metric  $g$  by  $dA_g$ , we show:

**Theorem 1.1.** *An oriented projective surface  $(M, \mathfrak{p})$  is metrisable by the metric  $g$  on  $M$  if and only if  $[g] : M \rightarrow (Z, J_{\mathfrak{p}})$  is a holomorphic curve and  $[g] : M \rightarrow (Z, \mathfrak{J}_{\mathfrak{p}, dA_g})$  is a pseudo-holomorphic curve.*

Applying a general existence result for pseudo-holomorphic curves [24, Theorem III] it follows that locally we can always find a Riemannian metric  $g$  so that  $[g] : M \rightarrow (Z, J_{\mathfrak{p}})$  is a holomorphic curve or so that  $[g] : M \rightarrow (Z, \mathfrak{J}_{\mathfrak{p}, dA_g})$  is a pseudo-holomorphic curve. The geometric significance of the existence of such (pseudo-)holomorphic curves is given in Proposition 2.8 below.

The construction of the (integrable) almost complex structure  $J_{\mathfrak{p}}$  on  $Z$  given in [7, 26] is adapted from the construction of an almost complex structure  $J$  on the twistor space  $Y \rightarrow N$  of an oriented Riemannian 4-manifold  $(N, g)$ , see [1]. In the Riemannian setting the almost complex structure  $J$  is integrable if and only if  $g$  is self-dual. In [12], Eells–Salamon observe that reversing the orientation on each fibre of  $Y \rightarrow N$  associates another almost complex structure  $\mathfrak{J}$  on  $Y$  to  $(N, g)$  which is never integrable. Thus, the non-integrable almost complex structure  $\mathfrak{J}$  used here may be thought of as the affine analogue of the non-integrable almost complex structure in oriented Riemannian 4-manifold geometry.

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## 2. Pseudo-Holomorphic Curves and Metrisability

Recall that the set of torsion-free connections on the tangent bundle of a surface  $M$  is an affine space modelled on the smooth sections of the vector bundle  $V = S^2(T^*M) \otimes TM$ . We have a natural trace mapping  $\text{tr} : V \rightarrow T^*M$ , given in abstract index notation by  $A_{jk}^i \mapsto A_{ik}^k$ , as well as an inclusion  $\text{Sym} : T^*M \rightarrow V$ , given by  $b_i \mapsto \delta_j^i b_k + \delta_k^i b_j$ . The bundle  $V$  thus decomposes as  $V = V_0 \oplus T^*M$ , where  $V_0$  denotes the trace-free part of  $V$ . We have (Cartan, Eisenhart, Weyl) – the reader may also consult [9] for a modern reference:

**Lemma 2.1.** *Two torsion-free connections  $\nabla$  and  $\nabla'$  on  $TM$  are projectively equivalent if and only if there exists a 1-form  $\xi$  on  $M$  so that  $\nabla - \nabla' = \text{Sym}(\xi)$ .*

This gives immediately:

**Lemma 2.2.** *Let  $(M, \mathfrak{p})$  be an oriented projective surface and  $\sigma$  a volume form on  $M$ . Then there exists a unique representative connection  ${}^\sigma\nabla \in \mathfrak{p}$  preserving  $\sigma$ .*

*Proof.* Let  $\nabla \in \mathfrak{p}$  be a representative connection. Since  $\sigma$  is a volume form there exists a unique 1-form  $\alpha$  on  $M$  such that  $\nabla\sigma = \alpha \otimes \sigma$ . An elementary computation shows that the connection  $\nabla + \text{Sym}(\xi)$  satisfies

$$(\nabla + \text{Sym}(\xi))\sigma = \nabla\sigma - 3\xi \otimes \sigma,$$

for all  $\xi \in \Omega^1(M)$ . Thus the connection  ${}^\sigma\nabla = \nabla + \frac{1}{3}\text{Sym}(\alpha)$  preserves  $\sigma$  and clearly is the only connection in  $\mathfrak{p}$  doing so.  $\square$

We also have:

**Lemma 2.3.** *Let  $\varphi \in \Gamma(V_0)$  and  $\nabla$  be a torsion-free connection on  $TM$ . Then  $\nabla + \varphi$  preserves a volume form  $\sigma$  on  $M$  if and only if  $\nabla$  preserves the volume form  $\sigma$ .*

*Proof.* Since  $\varphi \in \Gamma(V_0)$ , an elementary computation shows that the connections  $\nabla$  and  $\nabla + \varphi$  induce the same connection on the bundle  $\Lambda^2(T^*M)$  whose non-vanishing sections are the volume forms.  $\square$

For our purposes it is convenient to construct the almost complex structures  $(J, \mathfrak{J})$  associated to  $\nabla$  in terms of the connection form  $\theta$  on the oriented frame bundle of  $M$ . The oriented frame bundle  $F$  of the oriented surface  $M$  is the bundle  $\nu : F \rightarrow M$  whose fibre at  $x \in M$  consists of the linear isomorphisms  $u : \mathbb{R}^2 \rightarrow T_x M$  that are orientation preserving with respect to the standard orientation on  $\mathbb{R}^2$  and the given orientation on  $T_x M$ . The group  $\text{GL}^+(2, \mathbb{R})$  acts transitively from the right on each fibre by the rule  $R_a(u) = u \circ a$  for all  $a \in \text{GL}^+(2, \mathbb{R})$ ,  $u \in F$  and this action turns  $\nu : F \rightarrow M$  into a principal right  $\text{GL}^+(2, \mathbb{R})$ -bundle. The total space  $F$  carries a tautological  $\mathbb{R}^2$ -valued 1-form  $\omega$  defined by  $\omega_u = u^{-1} \circ v'_u$  and  $\omega$  satisfies the equivariance property

$$(2.1) \quad R_a^* \omega = a^{-1} \omega$$

for all  $a \in \text{GL}^+(2, \mathbb{R})$ . We may embed  $\text{GL}(1, \mathbb{C})$  as the subgroup of  $\text{GL}^+(2, \mathbb{R})$  consisting of matrices that commute with the standard linear complex structure on  $\mathbb{R}^2$ . Note that may think of the oriented frame bundle  $\nu : F \rightarrow M$  as a principal  $\text{GL}(1, \mathbb{C})$ -bundle over  $Z = F/\text{GL}(1, \mathbb{C})$ . We may describe an almost complex structure on  $Z$  by describing the pullback of its  $(1,0)$ -forms to  $F$ . The pullback of a 1-form on  $Z$  to  $F$  is *semi-basic* for the projection  $\nu : F \rightarrow Z$ , that is, it vanishes when evaluated on vector fields that are tangent to the fibres of  $\nu$ . For  $y \in \mathfrak{gl}(2, \mathbb{R})$  we denote by  $Y_y$  the vector field on  $F$  that is generated by the flow  $R_{\exp(ty)}$ . Clearly, the vector fields  $Y_y$  for  $y \in \mathfrak{gl}(1, \mathbb{C})$  span the vector fields on  $F$  that are tangent to the fibres of  $\nu$ .

Let  $\nabla$  be a torsion-free connection on  $TM$  with connection form  $\theta = (\theta_j^i)$  on  $F$ . Recall that  $\theta$  satisfies the equivariance property

$$(2.2) \quad R_a^* \theta = a^{-1} \theta a$$

for all  $a \in \mathrm{GL}^+(2, \mathbb{R})$  and the structure equations

$$(2.3) \quad \begin{aligned} d\omega^i &= -\theta_j^i \wedge \omega^j, \\ d\theta_j^i &= -\theta_k^i \wedge \theta_j^k + \Theta_j^i, \end{aligned}$$

where  $\Theta = (\Theta_j^i)$  denotes the curvature form of  $\theta$ . Since  $\theta$  is a principal connection on  $F$  it also satisfies  $\theta(Y_y) = y$  for all  $y \in \mathfrak{gl}(2, \mathbb{R})$ . Since the Lie algebra of  $\mathrm{GL}(1, \mathbb{C})$  is spanned by the matrices of the form

$$\begin{pmatrix} z & -w \\ w & z \end{pmatrix}$$

for  $(z, w) \in \mathbb{R}^2$ , the complex-valued 1-forms on  $F$  that are semi-basic for the projection  $\nu : F \rightarrow Z$  are spanned by the forms  $\omega = \omega^1 + i\omega^2$  and

$$\zeta = (\theta_1^1 - \theta_2^2) + i(\theta_2^1 + \theta_1^2)$$

and their complex conjugates. We now have:

**Proposition 2.4.** *Let  $\nabla$  be a torsion-free connection on  $TM$  with connection form  $\theta = (\theta_j^i)$  on  $F$ . Then there exists a unique pair  $(J, \mathfrak{J})$  of almost complex structures on  $Z$  whose  $(1,0)$ -forms pull back to become linear combinations of the forms  $(\omega, \zeta)$  in the case of  $J$  and to  $(\omega, \bar{\zeta})$  in the case of  $\mathfrak{J}$ . Moreover, the almost complex structure  $J$  is always integrable, whereas  $\mathfrak{J}$  is never integrable.*

*Proof.* Writing

$$re^{i\phi} \simeq \begin{pmatrix} r \cos \phi & -r \sin \phi \\ r \sin \phi & r \cos \phi \end{pmatrix}$$

for the elements of  $\mathrm{GL}(1, \mathbb{C})$ , the equivariance property (2.1) of  $\omega$  and (2.2) of  $\theta$  implies

$$(2.4) \quad (R_{re^{i\phi}})^* \omega = \frac{1}{r} e^{i\phi} \omega \quad \text{and} \quad (R_{re^{i\phi}})^* \zeta = e^{-2i\phi} \zeta.$$

It follows that there exists a unique almost complex structure  $J$  on  $Z$  whose  $(1,0)$ -forms pull back to  $F$  to become linear combinations of the forms  $\omega, \zeta$ . Likewise there exists a unique almost complex structure  $\mathfrak{J}$  on  $Z$  whose  $(1,0)$ -forms pull back to  $F$  to become linear combinations of the forms  $\omega, \bar{\zeta}$ . Furthermore, simple computations using the structure equations (2.3) imply that

$$0 = d\zeta \wedge \omega \wedge \zeta = d\omega \wedge \omega \wedge \zeta.$$

Consequently, the Newlander-Nirenberg theorem [23] implies that  $J$  is integrable. On the other hand, we get

$$d\omega \wedge \omega \wedge \bar{\zeta} = \frac{1}{2} \omega \wedge \bar{\omega} \wedge \zeta \wedge \bar{\zeta}$$

so that  $\mathfrak{J}$  is never integrable. □

**Remark 2.5.** The equivariance properties (2.4) imply that the bundles

$$H = \nu' \{ \mathrm{Re}(\zeta) = 0, \mathrm{Im}(\zeta) = 0 \} \quad \text{and} \quad V = \nu' \{ \mathrm{Re}(\omega) = 0, \mathrm{Im}(\omega) = 0 \}$$

are well-defined distributions on  $Z$  that are invariant with respect to  $J$  (and  $\mathfrak{J}$ ). Hence we have  $TZ = H \oplus V$ .

For the convenience of the reader, we also show [7, 26]:

**Proposition 2.6.** *Suppose the torsion-free connections  $\nabla$  and  $\nabla'$  on  $TM$  are projectively equivalent, then they induce the same integrable almost complex structure  $J$  on  $Z$ .*

*Proof.* The connections  $\nabla$  and  $\nabla'$  are projectively equivalent if and only if there exists a 1-form  $\xi$  on  $M$  such that  $\nabla' = \nabla + \text{Sym}(\xi)$ . Writing  $\theta = (\theta_j^i)$  for the connection form of  $\nabla$  on  $F$  and  $v^*\xi = x_i\omega^i$  for real-valued functions  $x_i$  on  $F$ , the connection form  $\theta'$  of  $\nabla'$  becomes

$$\theta' = \theta + \begin{pmatrix} 2x_1\omega^1 + x_2\omega^2 & x_2\omega^1 \\ x_1\omega^2 & x_1\omega^1 + 2x_2\omega^2 \end{pmatrix}.$$

Consequently, we obtain

$$\zeta' = \zeta + (x_1\omega^1 - x_2\omega^2) + i(x_2\omega^1 + x_1\omega^2) = \zeta + (x_1 + ix_2)\omega$$

which shows that the complex span of  $\omega, \zeta$  is the same as the one of  $\omega, \zeta'$  and hence the two integrable almost complex structures are the same.  $\square$

**Remark 2.7.** For a projective structure  $\mathfrak{p}$  on  $M$  we will write  $J_{\mathfrak{p}}$  for the integrable almost complex structure defined by any representative connection  $\nabla \in \mathfrak{p}$ . For a projective structure  $\mathfrak{p}$  and a volume form  $\sigma$  on  $M$  we will write  $\mathfrak{J}_{\mathfrak{p},\sigma}$  for the non-integrable almost complex structure defined by the representative connection  ${}^\sigma\nabla \in \mathfrak{p}$ . Note that the non-integrable almost complex structure is not projectively invariant.

Recall that a Weyl connection for a conformal structure  $[g]$  is a torsion-free connection  ${}^{[g]}\nabla$  on  $TM$  which preserves  $[g]$ . Fixing a Riemannian metric  $g \in [g]$ , the Weyl connections for  $[g]$  can be written as  ${}^{[g]}\nabla = {}^g\nabla + g \otimes B - \text{Sym}(\beta)$  for some 1-form  $\beta$  on  $M$  and where  $B$  denotes the  $g$ -dual vector field to  $\beta$ . In [21] and in the language of thermostats in [22], it was observed that for every choice of a conformal structure  $[g]$  on a projective surface  $(M, \mathfrak{p})$ , there exists a unique Weyl connection  ${}^{[g]}\nabla$  for  $[g]$  and a unique 1-form  $\varphi \in \Gamma(V_0)$  so that  ${}^{[g]}\nabla + \varphi$  is a representative connection of  $\mathfrak{p}$ . Moreover the endomorphism  $\varphi(X)$  is symmetric with respect to  $[g]$  for every vector field  $X$  on  $M$ . We call  ${}^{[g]}\nabla$  the Weyl connection determined by  $[g]$ . Explicitly, if  $\nabla$  is any representative connection of  $\mathfrak{p}$ ,  $g \in [g]$  and if we define a vector field  $B = \frac{3}{4}\text{tr} \left( g^\sharp \otimes (\nabla - {}^g\nabla)_0 \right)$ , then

$$\varphi = (\nabla - {}^g\nabla - g \otimes B)_0 \quad \text{and} \quad {}^{[g]}\nabla = {}^g\nabla + g \otimes B - \text{Sym}(\beta),$$

where  $A_0$  denotes the trace-free part of a tensor field  $A \in \Gamma(S^2(T^*M) \otimes TM)$ . We refer the reader to [21, 22] for a proof that  ${}^{[g]}\nabla$  and  $\varphi$  do satisfy the claimed properties.

**Proposition 2.8.** *Let  $(M, \mathfrak{p})$  be an oriented projective surface and  $g$  a Riemannian metric on  $M$ . Then we have:*

- (i)  $\mathfrak{p}$  contains a Weyl connection for  $[g]$  if and only if  $[g] : M \rightarrow (Z, J_{\mathfrak{p}})$  is a holomorphic curve;

- (ii) *the Weyl connection determined by  $[g]$  is the Levi-Civita connection of  $g$  if and only if  $[g] : M \rightarrow (Z, \mathfrak{J}_p, dA_g)$  is a pseudo-holomorphic curve.*

**Remark 2.9.** Here we say  $[g] : M \rightarrow (Z, \mathfrak{J})$  is a (pseudo-)holomorphic curve if the image  $\Sigma = [g](M) \subset Z$  admits the structure of a (pseudo-)holomorphic curve. By admitting the structure of (pseudo-)holomorphic curve, we mean that  $\Sigma$  can be equipped with a complex structure  $J$ , so that the inclusion  $\iota : \Sigma \rightarrow Z$  is  $(J, \mathfrak{J})$ -linear, that is, satisfies  $\mathfrak{J} \circ \iota' = \iota' \circ J$ .

As an immediate consequence, we obtain the Theorem 1.1:

*Proof of Theorem 1.1.* The projective structure  $\mathfrak{p}$  is metrisable by  $g$  if and only if the Weyl connection determined by  $[g]$  is the Levi-Civita connection of  $g$  and the 1-form  $\varphi$  vanishes identically. The claim follows by applying Proposition 2.8.  $\square$

For the proof of Proposition 2.8 we also need the following Lemma:

**Lemma 2.10.** *Let  $(Z, \mathfrak{J})$  be an almost complex four-manifold and  $\omega, \chi \in \Omega^1(Z, \mathbb{C})$  a basis for the  $(1,0)$ -forms of  $Z$ . Suppose  $\iota : \Sigma \rightarrow Z$  is an immersed surface so that  $\iota^*(\omega \wedge \bar{\omega})$  is non-vanishing on  $\Sigma$ . Then  $\Sigma$  admits the structure of a pseudo-holomorphic curve if and only if  $\iota^*(\omega \wedge \chi)$  vanishes identically on  $\Sigma$ .*

*Proof.* Since  $\iota^*(\omega \wedge \bar{\omega})$  is non-vanishing on  $\Sigma$ , the forms  $\iota^*\omega$  and  $\iota^*\bar{\omega}$  span the complex-valued 1-forms on  $\Sigma$ . Recall that  $\iota : \Sigma \rightarrow Z$  is  $(j, \mathfrak{J})$ -linear if and only if the pullback of every  $(1,0)$ -form on  $Z$  is a  $(1,0)$ -form on  $\Sigma$ , the claim follows.  $\square$

*Proof of Proposition 2.8.* Let  $g$  be a Riemannian metric on the oriented projective surface  $(M, \mathfrak{p})$ . Without losing generality we can assume that the projective structure  $\mathfrak{p}$  arises from a connection of the form  ${}^{[g]}\nabla + \varphi$ . The Weyl connection  ${}^{[g]}\nabla$  satisfies

$${}^{[g]}\nabla dA_g = 2\beta \otimes dA_g$$

for some 1-form  $\beta$  on  $M$  and hence can be written as  ${}^{[g]}\nabla = {}^g\nabla + g \otimes \beta^\# - \text{Sym}(\beta)$ .

Now suppose  $\nabla \in \mathfrak{p}$  preserves the volume form  $dA_g$  of  $g$ . Then, by Lemma 2.3 it must be of the form

$$(2.5) \quad \nabla = {}^{[g]}\nabla + \varphi + \frac{2}{3}\text{Sym}(\beta) = {}^g\nabla + g \otimes \beta^\# - \frac{1}{3}\text{Sym}(\beta) + \varphi.$$

Proposition 2.4 and Lemma 2.10 imply that the condition that  $[g] : M \rightarrow Z$  defines a pseudo-holomorphic curve with respect to  $J_p$  respectively  $\mathfrak{J}_{p, dA_g}$  is equivalent to the condition that on the pullback bundle  $[g]^*F \rightarrow M$  the form  $\omega \wedge \zeta$ , respectively  $\omega \wedge \bar{\zeta}$  vanishes identically, where  $\zeta$  is computed from the connection form of  $\nabla$  and where we think of  $F$  as fibering over  $Z$ . Keeping this in mind we now compute the pullback of the forms  $\zeta$  and  $\bar{\zeta}$  to  $[g]^*F$ . Recall that the semi-basic 1-forms on  $F$  are spanned by the components

of  $\omega$ , hence there exist unique real-valued functions  $g_{ij} = g_{ji}$  on  $F$  so that  $v^*g = g_{ij}\omega^i \otimes \omega^j$ . Likewise, there exist unique real-valued functions  $b_i$  on  $F$  so that  $v^*\beta = b_i\omega^i$  and unique real-valued function  $A_{jk}^i = A_{kj}^i$  on  $F$  so that  $(v^*\varphi)_j^i = A_{jk}^i\omega^k$ . The functions  $A_{jk}^i$  satisfy furthermore  $A_{ki}^k = 0$  and  $g_{ik}A_{jl}^k = g_{jk}A_{il}^k$  since  $\varphi$  takes values in the endomorphisms of  $TM$  that are trace-free and symmetric with respect to  $g$ . The Levi-Civita connection  $(\psi_j^i)$  of  $g$  is the unique principal  $GL^+(2, \mathbb{R})$ -connection on  $F$  that satisfies

$$\begin{aligned} d\omega^i &= -\psi_j^i \wedge \omega^j, \\ dg_{ij} &= g_{ik}\psi_j^k + g_{kj}\psi_i^k. \end{aligned}$$

The pullback bundle  $P := [g]^*F$  is cut out by the equations  $g_{11} = g_{22}$  and  $g_{12} = 0$ . On  $P$  we have

$$\begin{aligned} 0 &= dg_{12} = g_{11}\psi_2^1 + g_{22}\psi_1^2 = g_{11}(\psi_2^1 + \psi_1^2), \\ 0 &= dg_{11} - dg_{22} = 2g_{11}\psi_1^1 - 2g_{22}\psi_2^2 = g_{11}(\psi_1^1 - \psi_2^2) \end{aligned}$$

On  $P$  the condition  $g_{ik}A_{jl}^k = g_{jk}A_{il}^k$  implies  $A_{11}^2 = -A_{22}^2$  and  $A_{22}^1 = -A_{11}^1$ . Writing  $A_{11}^1 = a_1$  and  $A_{22}^2 = a_2$  and using (2.5), the connection form  $\theta$  of  $\nabla$  thus becomes

$$\begin{aligned} \theta &= \begin{pmatrix} \psi_1^1 & -\psi_1^2 \\ \psi_2^1 & \psi_1^1 \end{pmatrix} + \begin{pmatrix} b_1\omega^1 & b_1\omega^2 \\ b_2\omega^1 & b_2\omega^2 \end{pmatrix} - \frac{1}{3} \begin{pmatrix} 2b_1\omega^1 + b_2\omega^2 & b_2\omega^1 \\ b_1\omega^2 & b_1\omega^1 + 2b_2\omega^2 \end{pmatrix} \\ &\quad + \begin{pmatrix} a_1\omega^1 - a_2\omega^2 & -a_2\omega^1 - a_1\omega^2 \\ -a_2\omega^1 - a_1\omega^2 & -a_1\omega^1 + a_2\omega^2 \end{pmatrix} \end{aligned}$$

Introducing the complex notation  $a = a_1 + ia_2$  and  $b = \frac{1}{2}(b_1 - ib_2)$ , we obtain from a simple calculation

$$\zeta = (\theta_1^1 - \theta_2^2) + i(\theta_2^1 + \theta_1^2) = \frac{4}{3}\bar{b}\omega + 2\bar{a}\bar{\omega},$$

where we write  $\omega = \omega^1 + i\omega^2$ .

Finally, since  $[g] : M \rightarrow (Z, J_p)$  is a holomorphic curve if and only if  $\omega \wedge \zeta$  vanishes identically on  $P$ , it follows that  $[g] : M \rightarrow (Z, J_p)$  is a holomorphic curve if and only if

$$0 = \omega \wedge \zeta = 2\bar{a}\omega \wedge \bar{\omega}$$

which is equivalent to  $\varphi$  vanishing identically. This shows (i).

Likewise  $[g] : M \rightarrow (Z, \mathfrak{I}_{p,dA_g})$  is a pseudo-holomorphic curve if and only if

$$0 = \omega \wedge \bar{\zeta} = \frac{4}{3}b\omega \wedge \bar{\omega}$$

on  $P$ . This is equivalent to  $\beta$  vanishing identically. This shows (ii).  $\square$

As a corollary we obtain:

**Corollary 2.11.** *Let  $(M, p)$  be a projective surface. Then locally  $p$  contains*

- (i) *a Weyl connection  ${}^{[g]}\nabla$  for some conformal structure  $[g]$ ;*
- (ii) *a connection of the form  $\tilde{g}\nabla + \varphi$  for some Riemannian metric  $\tilde{g}$  and some  $\varphi \in \Gamma(V_0)$  with  $\varphi$  taking values in the endomorphisms that are  $\tilde{g}$ -symmetric.*

**Remark 2.12.** The first statement of Proposition 2.8 and Corollary 2.11 was previously obtained in [19].

*Proof of Corollary 2.11.* We first consider the case (ii). We fix a volume form  $\sigma$  on  $M$ . We need to show that in a neighbourhood  $U_x$  of every point  $x \in M$  there exists a conformal structure  $[g]$  which is a pseudo-holomorphic curve into the total space of the bundle  $\pi : Z \rightarrow M$ , where we equip  $Z$  with the almost complex structure  $\mathfrak{J}_{p,\sigma}$ . Choose  $j \in Z$  with  $\pi(j) = x$ . Recall from Remark 2.5 that the subspace  $H_j \subset T_j Z$  is invariant under  $\mathfrak{J}_{p,\sigma}$ . Now [24, Theorem III] implies that there exists a pseudo-holomorphic curve  $\Sigma \subset (Z, \mathfrak{J}_{p,\sigma})$  which contains  $j$  and has  $H_j$  as its tangent space at  $j$ . Since  $H_j \subset T_j Z$  is horizontal, the restriction  $\pi'_j|_{H_j} : H_j \rightarrow T_x M$  is an isomorphism. Therefore, the restriction of  $\pi$  to  $\Sigma$  is a local diffeomorphism in some neighbourhood of  $j$ . Hence there exists a neighbourhood  $U_x$  of  $x \in M$  and a section  $[g] : U_x \rightarrow Z$  so that  $[g](U_x) \subset \Sigma$ . Thus,  $[g] : U_x \rightarrow (Z, \mathfrak{J}_{p,\sigma})$  is a pseudo-holomorphic curve in the sense of Remark 2.9. Taking  $\tilde{g}$  to be the unique metric in  $[g]$  with volume form  $\sigma$  and applying Proposition 2.8 shows the claim. The case (i) follows in the same fashion, except that [24] is not needed, as  $J_p$  is integrable and hence the construction of a holomorphic curve realising a prescribed  $J_p$ -invariant tangent plane is an elementary exercise.  $\square$

**Remark 2.13.** Locally we can always find a holomorphic curve  $[g] : M \rightarrow (Z, J_p)$ , but globally this is not always possible. A properly convex projective structure  $p$  on a closed surface  $M$  with  $\chi(M) < 0$  admits a holomorphic curve  $[g] : M \rightarrow (Z, J_p)$  if and only if  $p$  is hyperbolic [22]. One would expect that a corresponding global non-existence result should also hold in the pseudo-holomorphic setting for a suitable class of projective surfaces.

**Remark 2.14.** If  $(M, p)$  is a closed oriented projective surface of with  $\chi(M) < 0$ , then there exists at most one holomorphic curve  $[g] : M \rightarrow (Z, J_p)$ , see [20].

**Remark 2.15.** Hitchin [15] gave a twistorial construction of (complex) two-dimensional holomorphic projective structures. In the holomorphic category such a projective structure corresponds to a complex surface  $Z$  having a family of rational curves with self-intersection number one. Denoting the canonical bundle of  $Z$  by  $K_Z$ , such a holomorphic projective surface is metrisable if and only if  $K_Z^{-2/3}$  admits a holomorphic section which intersects each rational curve in  $Z$  at two points [2, 3, 16].

**Remark 2.16.** The notion of a projective structure also makes sense in the complex setting and such structures are referred to as *c-projective*, see [4]. Correspondingly, there is a Kähler metrisability problem of c-projective structures. Some obstructions to Kähler metrisability of a (complex) two-dimensional c-projective structure have been obtained in [18].

We conclude by describing the holomorphic curves for the standard projective structure  $p_0$  on the 2-sphere whose geodesics are the great circles.

**Example 2.17.** Let  $S^2$  denote the sphere of radius 1 centred at the origin in  $\mathbb{R}^3$  and  $g$  its induced round metric of constant Gauss curvature 1 whose geodesics are the great circles. We equip  $S^2$  with its standard orientation.

Recall that the unit tangent bundle  $\lambda : T_1 S^2 \rightarrow S^2$  of  $(S^2, g)$  carries a canonical coframing  $(\omega_1, \omega_2, \psi)$ , where  $\omega_1, \omega_2$  span the 1-forms on  $T_1 S^2$  that are semi-basic for the projection  $\lambda$  and  $\psi$  denotes the Levi-Civita connection form of  $g$ . The 1-forms  $(\omega_1, \omega_2, \psi)$  satisfy the structure equations

$$(2.6) \quad d\omega_1 = -\omega_2 \wedge \psi \quad \text{and} \quad d\omega_2 = -\psi \wedge \omega_1 \quad \text{and} \quad d\psi = -\omega_1 \wedge \omega_2.$$

Let  $\hat{g}$  be a Riemannian metric on  $S^2$  and write  $\lambda^* \hat{g} = \hat{g}_{ij} \omega_i \otimes \omega_j$  for unique real-valued functions  $\hat{g}_{ij} = \hat{g}_{ji}$  on  $T_1 S^2$ . Phrased in modern language (c.f. [2]) and applied to the case of the 2-sphere, R. Liouville's result [17] implies that if the metrics  $\hat{g}$  and  $g$  have the same unparametrised geodesics then the functions  $h_{ij} := \hat{g}_{ij}(\hat{g}_{11}\hat{g}_{22} - \hat{g}_{12}^2)^{-2/3}$  satisfy the linear differential equations

$$(2.7) \quad \begin{aligned} dh_{11} &= -2h_{11}\omega_2 + 2h_{12}\psi, \\ dh_{12} &= h_{11}\omega_1 - h_{22}\omega_2 - (h_{11} - h_{22})\psi, \\ dh_{22} &= 2h_2\omega_1 - 2h_{12}\psi, \end{aligned}$$

for some smooth real-valued functions  $h_i$  on  $T_1 S^2$ . Conversely, a solution to (2.7) on  $T_1 S^2$  satisfying  $h_{11}h_{22} - h_{12}^2 \neq 0$  gives a Riemannian metric  $\hat{g}$  on  $S^2$  with  $\lambda^* \hat{g} = (h_{ij}(h_{11}h_{22} - h_{12}^2)^{-2})\omega_i \otimes \omega_j$  and that has the same unparametrised geodesics as  $g$ .

Applying the exterior derivative to the above system of equations implies the existence of a unique real-valued function  $h$  on  $T_1 S^2$  such that

$$\begin{aligned} dh_1 &= -h_{12}\omega_1 + (h_{11} + h)\omega_2 + h_2\psi, \\ dh_2 &= -(h_{22} + h)\omega_1 + h_{12}\omega_2 - h_1\psi. \end{aligned}$$

Taking yet another exterior derivative gives that

$$dh = -2h_1\omega_1 + 2h_2\omega_2.$$

Writing

$$\vartheta = \begin{pmatrix} 0 & -\omega_1 & -\omega_2 \\ \omega_1 & 0 & -\psi \\ \omega_2 & \psi & 0 \end{pmatrix} \quad \text{and} \quad H = \begin{pmatrix} h & h_2 & -h_1 \\ h_2 & -h_{22} & h_{12} \\ -h_1 & h_{12} & -h_{11} \end{pmatrix}$$

the above system of differential equations can be expressed as

$$dH + \vartheta H + H\vartheta^t = 0.$$

The structure equations (2.6) imply that  $d\vartheta + \vartheta \wedge \vartheta = 0$ , hence we may write  $\vartheta = \Xi^{-1}d\Xi$  for some diffeomorphism  $\Xi : T_1 S^2 \rightarrow \text{SO}(3)$ . It follows that the solutions are of the form  $H = \Xi^{-1}C(\Xi^{-1})^t$  for some constant symmetric 3-by-3 matrix  $C$ . In particular, taking  $C = AA^t$  for some  $A \in \text{SL}(3, \mathbb{R})$ , we obtain a solution  $H_A$  providing a metric  $\hat{g}_A$  on  $S^2$  having the great circles as its geodesics.

Finally, in order to construct the holomorphic curve  $[\hat{g}_A] : S^2 \rightarrow Z$  from  $H_A$ , we interpret  $Z$  as an associated bundle to  $T_1 S^2$ . We will only give a sketch of the construction and refer the reader to [22, §4] for additional details. The orientation and metric turn  $S^2$  into a Riemann surface and hence a conformal structure on  $S^2$  is given in terms of a Beltrami differential. Denoting the canonical bundle of  $S^2$  by  $K_{S^2}$ , a Beltrami differential is a section  $\mu$  of  $\overline{K_{S^2}} \otimes K_{S^2}^{-1}$  satisfying  $|\mu(x)| < 1$  for all  $x \in S^2$ , where  $|\cdot|$  denotes the norm induced by the natural Hermitian bundle metric on  $\overline{K_{S^2}} \otimes K_{S^2}^{-1}$ . The Riemannian metric  $g$  gives an isomorphism  $\overline{K_{S^2}} \otimes K_{S^2}^{-1} \simeq K_{S^2}^{-2}$  and thus  $Z$  may be identified with  $T_1 S^2 \times_{S^1} \mathbb{D}$ , where  $S^1$  acts by usual rotation on  $T_1 S^2$  and by  $z \cdot e^{i\phi} = ze^{-2i\phi}$  on the open unit disk  $\mathbb{D} \subset \mathbb{C}$ . A holomorphic curve  $[\hat{g}] : S^2 \rightarrow Z$  is therefore represented by a map  $\mu : T_1 S^2 \rightarrow \mathbb{D}$ . Explicitly, the conformal structure arising from a Riemannian metric  $\hat{g}$  on  $S^2$  is represented by the map

$$\mu = \frac{p - q + 2ir}{p + q + 2\sqrt{pq - r^2}},$$

where we write  $\lambda^* \hat{g} = p\omega_1 \otimes \omega_1 + 2r\omega_1 \otimes \omega_2 + q\omega_2 \otimes \omega_2$  for unique real-valued functions  $p, q, r$  on  $T_1 S^2$ . In our case, the holomorphic curve  $[\hat{g}_A] : S^2 \rightarrow Z$  is thus represented by  $\mu$  with

$$p = \frac{h_{11}}{(h_{11}h_{22} - h_{12}^2)^2}, \quad r = \frac{h_{12}}{(h_{11}h_{22} - h_{12}^2)^2}, \quad q = \frac{h_{22}}{(h_{11}h_{22} - h_{12}^2)^2}$$

and where the functions  $h_{ij}$  arise from  $H_A$  as above.

**Remark 2.18.** In the case of the standard projective structure on  $S^2$  the complex surface  $(Z, J_{p_0})$  is biholomorphic to  $\mathbb{CP}^2 \setminus \mathbb{RP}^2$  and moreover, the image of a holomorphic curve  $[g] : S^2 \rightarrow Z$  is a smooth quadric, see [19]. Trying to explicitly relate the holomorphic curve  $[\hat{g}_A]$  to its image quadric does in general however not seem to give manageable expressions.

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